Pergamon

PII: S0021-8928(98)00081-1

J. Appl. Maths Mechs, Vol. 62, No. 4, pp. 641–648, 1998

0021-8928/98/$—see front matter

HOMOGENEOUS SOLUTIONS AND SAINT-VENANT PROBLEMS FOR A HELICAL SPRING†

D. P. KARGIN, N. V. KURBATOVA and Yu. A. USTINOV

Rostov-on-Don

(*Received* 22 *May* 1997)

The three-dimensional problem of the theory of elasticity for a spring with a stress-free side surface is investigated. In [1] the problem was reduced to an eigenvalue problem on a section, which enables a complete system of homogeneous elementary solutions to be constructed, and a group of 12 elementary solutions were distinguished, on the basis of which the construction of a Saint-Venant solution was reduced to two types of two-dimensional problems and an algebraic system of equations in the coefficients of the expansion. A variational formulation of these problems is given and the results of an asymptotic and numerical investigation of all solutions and of the stiffness matrix are presented. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM AND THE HOMOGENEOUS SOLUTIONS FOR A SPRING

The spring will be regarded as a three-dimensional elastic body, which is obtained by the helical motion of a plane figure S, situated in the ϕ = const plane of a cylindrical system of coordinates r, ϕ, z. We will use the following notation: r_0 is the distance from the z axis to the centre of gravity of the figure S, h is the pitch of the spring, μ is the shear modulus and ν is Poisson's ratio.

We will introduce a new system of coordinates, connected with the cylindrical system by the following relations

$$\begin{gathered}\xi = r - r_0,\quad \xi_2 = z - \xi h_0,\quad \xi_3 = \xi = \phi + 2\pi(m-1)\\ \xi_1,\ \xi_2 \in S,\ \xi \in [0,\eta],\ \eta = 2\pi n + \alpha\ \ (0 \leqslant \alpha < 2\pi),\ h_0 = h/(2\pi),\ m = 1,\dots,n\end{gathered}\tag{1.1}$$

Here n is the number of turns in the section ϕ = const $(0 \leqslant \phi \leqslant 2\pi)$, the variable ξ defines the section of the spring, while ξ_1, ξ_2 is a point in the chosen section, and the values $\xi = 0$, $\xi = \eta$ correspond to the end sections of the spring.

In the new system of coordinates $V = S \times [0, \eta]$ is the region occupied by the spring and $\Gamma = \partial S \times [0, \eta]$ are the side surface, where ∂S is the boundary of S. The projections of the outward unit normal $\mathbf{N}$ on to the unit normal of the cylindrical system of coordinates are related to the projections of the unit normal to S as follows:

$$N_r = N_1 = \frac{n_1}{w},\quad N_z = N_2 = \frac{n_2}{w},\quad N_\phi = N_3 = -f\frac{n_2}{w};\quad w = (1 + f^2 n_2^2)^{1/2},\quad f = h_0 r \tag{1.2}$$

We will now consider the fundamental relations of the theory of elasticity. If we take into account that

$$\nabla = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}D,\quad \partial_\alpha = \frac{\partial}{\partial\xi_\alpha},\quad D = \frac{1}{r}(\partial - h_0\partial_2),\quad \partial = \frac{\partial}{\partial\xi},\quad \alpha = 1,2$$

we obtain the following expressions for the components of the strain tensor

$$\begin{gathered}e_{11} = \partial_1 u_1,\quad e_{22} = \partial_2 u_2,\quad e_{33} = Du_3 + u_1/r\\ 2e_{12} = \partial_1 u_2 + \partial_2 u_1,\quad 2e_{13} = Du_1 + \partial_1 u_3 - u_3/r,\quad 2e_{23} = Du_2 + \partial_2 u_3\end{gathered}\tag{1.3}$$

Here and below $\mathbf{e}_1 = \mathbf{e}_r$, $\mathbf{e}_2 = \mathbf{e}_z$, $\mathbf{e} = \mathbf{e}_3 = \mathbf{e}_\phi$, where $\mathbf{e}_r$, $\mathbf{e}_z$ and $\mathbf{e}_\varphi$ are the unit vectors of a cylindrical system of coordinates and u_k are the components of the displacement vector $\mathbf{u}$.

†*Prikl. Mat. Mekh.* Vol. 62, No. 4, pp. 690–698, 1998.

Note 1. The fundamental relations, such as the components of the strain tensor and the equilibrium equations in the stresses and displacements, can be obtained from the corresponding expressions in a cylindrical system of coordinates [2] by replacing the operator $r^{-1}\partial$ by the operator D.

The object of the investigation is the following problem of the theory of elasticity: It is required to find the solution of the equations of the theory of elasticity in the region V, which satisfies the following boundary conditions

$$N_i\sigma_{ij}|_\Gamma = 0 \tag{1.4}$$

$$u_i|_{\xi=0} = 0, \quad \sigma_{3i}|_{\xi=\eta} = p_i \tag{1.5}$$

Here and blow σ_{ij} are the components of the strain tensor, p_i are the specified external stresses, Latin subscripts take the values 1, 2, and 3, Greek subscripts take the values 1 and 2, and summation is carried out over repeated subscripts.

Basing ourselves on boundary conditions (1.4) and (1.5), we will call this problem the Saint-Venant problem for a spring.

In [1] the problem was reduced to the following operator equation

$$\begin{gathered} L(\partial)\mathbf{u} = \{L(\partial)\mathbf{u}, M(\partial)\mathbf{u}\} = 0 \\ L(\partial)\mathbf{u} = \partial^2 C\mathbf{u} + \partial B\mathbf{u} + A\mathbf{u}, \quad M(\partial)\mathbf{u} = (\partial G\mathbf{u} + E\mathbf{u})|_\Gamma \end{gathered} \tag{1.6}$$

and the equivalent eigenvalue problem

$$L(\gamma)\mathbf{a} = 0 \tag{1.7}$$

provided that $\mathbf{u} = \mathbf{a}(\xi_1, \xi_2)e^{\gamma\xi}$.

Relations (1.6) symbolize the equilibrium equations and the boundary conditions on the side surface Γ, and A, B, C, G, E are differential operators with respect to the variables ξ_1, ξ_2, the specific form of which can easily be established on the basis of note 1.

Basing ourselves on the consideration that the components of the vector of the rigid displacement of a spring can be represented by the following expressions

$$\begin{aligned} u_1^0 &= C_3e^{i\xi} + C_4e^{-i\xi} + C_5(\xi + h_0^{-1}\xi_2)e^{i\xi} + C_6(\xi + h_0^{-1}\xi_2)e^{-i\xi} \\ u_2^0 &= C_1 - C_5h_0^{-1}re^{i\xi} - C_6h_0^{-1}re^{-i\xi} \\ u_3^0 &= iC_3e^{i\xi} - iC_4e^{-i\xi} + iC_5(\xi + h_0^{-1}\xi_2)e^{i\xi} - iC_6(\xi + h_0^{-1}\xi_2)e^{-i\xi} + C_2r / r_0 \\ C_1 &= a_z, \; C_2 = \omega_z r_0, \; C_3 = (a_x - ia_y)/2, \; C_4 = (a_x + ia_y)/2 \\ C_5 &= h_0(\omega_y + i\omega_x)/2, \quad C_6 = h_0(\omega_y - i\omega_x)/2 \end{aligned} \tag{1.8}$$

we can conclude that $\gamma = 0, i, -i$ are the eigenvalues of eigenvalue problem (1.7), where $\gamma = 0$ corresponds to the pair of eigenvectors

$$\mathbf{a}_1 = (0, 1, 0), \quad \mathbf{a}_2 = (0, 0, r/r_0)$$

$\gamma = i$ correspond to the eigenvector and associated vector

$$\mathbf{a}_3 = (1, 0, i), \quad \mathbf{a}_5 = r_0^{-1}(\xi_2, -r, i\xi_2)$$

and $\gamma = -i$ correspond to the eigenvector and associated vector

$$\mathbf{a}_4 = \mathbf{a}_3^*, \quad \mathbf{a}_6 = \mathbf{a}_5^*$$

where $\mathbf{a}^*$ is the complex conjugate of $\mathbf{a}$. In expressions (1.8) a_x^0, a_y^0, a_z^0 are the components of the vector $\mathbf{a}^0$ of the translational displacement in a Cartesian system of coordinates and ω_x, ω_x,ω_z are the components of the vector ω of the angle of small rotation.

However, the system of eigenvectors and associated vectors given above does not exhaust the root subspaces of these eigenvalues. The eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ have a single associated vector, namely $\mathbf{a}_7$ and $\mathbf{a}_8$ respectively, which are solutions of the boundary-value problems

$$A_h\mathbf{a}_7 = 0,\quad E_h\mathbf{a}_7|_{\partial S} = (0, n_2 f, -n_2) \tag{1.9}$$

$$A_h\mathbf{a}_8 = (2,0,0),\quad E_h\mathbf{a}_8|_{\partial S} = (-2\theta n_1, -2\theta n_2, 2(1+\theta) f n_2),\quad \theta = \nu/(1-2\nu) \tag{1.10}$$

The eigenvector $\mathbf{a}_3$, in addition to the associated vector $\mathbf{a}_5$, has two more associated vectors $\mathbf{a}_9$ and $\mathbf{a}_{11}$, which are solutions of the following boundary-value problems

$$L_h(i)\mathbf{a}_\beta = F_\beta,\quad M_h(i)\mathbf{a}_\beta = \mathbf{f}_\beta\quad (\beta = 9, 11) \tag{1.11}$$

$$F_9 = -(iC_h + B_h)\mathbf{a}_3,\quad \mathbf{f}_9 = -G_h\mathbf{a}_3,\quad F_{11} = -(iC_h + B_h)\mathbf{a}_9 - 2C_h\mathbf{a}_3,\quad \mathbf{f}_{11} = -G_h\mathbf{a}_9$$

The eigenvector $\mathbf{a}_4$, in addition to the associated vector $\mathbf{a}_6$, has two more associated vectors, namely $\mathbf{a}_{10} = \mathbf{a}_9^*$ and $\mathbf{a}_{12} = \mathbf{a}_{11}^*$.

The specific form of the vectors F_β, f_β was derived previously in [1], so below we will give the variational formulation of problem (1.9)–(1.11).

The system of eigenvectors and associated vectors described above defines 12 linearly independent elementary solutions

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{a}_1,\quad \mathbf{u} = \mathbf{a}_2,\quad \mathbf{u}_3 = e^{i\xi}\mathbf{a}_3,\quad \mathbf{u}_4 = \mathbf{u}_3^*,\quad \mathbf{u}_5 = e^{i\xi}(\xi\mathbf{a}_3 + \mathbf{a}_5),\quad \mathbf{u}_6 = \mathbf{u}_5^* \\
&\mathbf{u}_7 = \xi\mathbf{a}_1 + \mathbf{a}_7,\quad \mathbf{u}_8 = \xi\mathbf{a}_2 + \mathbf{a}_8,\quad \mathbf{u}_9 = e^{i\xi}(\tfrac{1}{2}\xi^2\mathbf{a}_3 + \xi\mathbf{a}_5 + \mathbf{a}_9),\quad \mathbf{u}_{10} = \mathbf{u}_9^* \\
&\mathbf{u}_{11} = e^{i\xi}(\tfrac{1}{6}\xi^3\mathbf{a}_3 + \tfrac{1}{2}\xi^2\mathbf{a}_5 + \xi\mathbf{a}_9 + \mathbf{a}_{11}),\quad \mathbf{u}_{12} = \mathbf{u}_{11}^*
\end{aligned} \tag{1.12}$$

The remaining part of the set of eigenvalues Λ_p possesses the following property.

Property 1. Among the eigenvalues $\gamma_k \in \Lambda_p$ there is none that is pure imaginary.

This property can be proved using the same methods that were employed to prove the analogous property for a naturally twisted rod [3].

We will denote the stress vector in the section ξ = const by $\sigma(\sigma_{13}, \sigma_{23}, \sigma_{33})$. Basing on the theorem on the completeness of a system of elementary solutions [4], the solution of problem (1.6) can be represented in the form

$$\mathbf{u} = \mathbf{u}_s + \mathbf{u}_p,\quad \sigma = \sigma_s + \sigma_p \tag{1.13}$$

$$\begin{aligned}
&\mathbf{u}_s = \sum_{l=1}^{6}[C_l\mathbf{u}_l(\xi) + C_{6+l}\mathbf{u}_{6+l}(\xi - \eta)],\quad \sigma_s = \sum_{l=1}^{6} C_{l+6}\boldsymbol{\sigma}_{l+6}(\xi - \eta) \\
&\mathbf{u}_p = \sum_k C_k\mathbf{u}_k(\xi),\quad \sigma_p = \sum_{l=1}^{6} C_j\sigma_j(\xi - \eta)
\end{aligned} \tag{1.14}$$

where C_t are arbitrary constants, which are found from the infinite systems obtained when boundary conditions (1.5) are satisfied (a method of constructing such systems was presented in [4, 5])

$$\sigma_m = 0,\ m = 1, \ldots, 6$$

$$\begin{aligned}
&\sigma_7 = \mu\mathbf{b}_1,\quad \sigma_8 = \mu\mathbf{b}_2,\quad \sigma_9 = \mu\mathbf{b}_3 e^{i\xi},\quad \sigma_{10} = \mu\sigma_9^* \\
&\sigma_{11}(\xi) = \mu(\xi\mathbf{b}_3 + \mathbf{b}_5)e^{i\xi},\quad \sigma_{12}(\xi) = \sigma_{11}^*(\xi)
\end{aligned}$$

The components of the vectors b_{km} (the first subscript corresponds to the projection while the second corresponds to the number of the vector) are defined by the expressions

$$\begin{aligned}
&b_{1m} = D_m a_{16+m} + \partial_1 a_{36+m} - a_{36+m}/r + b_{1m}^0,\quad b_{2m} = D_m\partial_2 a_{26+m} + \partial_2 a_{36+m} + b_{2m}^0 \\
&b_{3m} = 2[(1+\theta)(D_m a_{36+m} + a_{16+m}/r) + \partial_1 a_{16+m} + \partial_2 a_{26+m}] + b_{3m}^0 \\
&b_{11}^0 = 0,\quad b_{21}^0 = 1/r,\quad b_{31}^0 = 0,\quad b_{12}^0 = 0,\quad b_{22}^0 = 0,\quad b_{32}^0 = 2(1+\theta)/r_0 \\
&b_{13}^0 = b_{14}^0 = \xi_2/(rr_0),\quad b_{23}^0 = b_{14}^0 = -1/r_0,\quad b_{33}^0 = b_{34}^{0*} = 2(1+\theta)i\xi_2/(rr_0)
\end{aligned} \tag{1.15}$$

$$b_{15}^0 = b_{16}^{0^*} = a_{19} / r, \quad b_{25}^0 = b_{26}^{0^*} = a_{29} / r, \quad b_{35}^0 = b_{36}^{0^*} = 2(1+\theta)a_{39} / r$$
$$D_m = h_0\partial_2 \ (m = 1,2), \quad D_m = h_0\partial_2 + i \ (m = 3,5), \quad D_m = h_0\partial_2 - i \ (m = 4,6)$$

From the generalized orthogonality relations we have

$$(\sigma_k(\xi), \mathbf{a}_l) = 0, \quad l = 1,\dots,6$$
$$(\mathbf{b}, \mathbf{a}) = \int \mathbf{b} * \mathbf{a}^* dS = \int b_{jk} a_{jk}^* dS, \quad j = 1,2,3 \tag{1.16}$$

Here and everywhere later the integration is carried out over the section S.

By virtue of property 1 the terms $\mathbf{u}_p$ and σ_p define the boundary layer in the neighbourhood of the spring ends $\xi = 0$, η, and by virtue of relations (1.16) the stress state corresponding to it is self-balanced in any section ξ = const. Hence, it is natural to call $\mathbf{u}_s$ the Saint-Venant solution for the spring. Unlike a cylinder, in this case all the components of the stress tensor, defined with respect to the vector $\mathbf{u}_s$, are non-zero, and it is this that causes the main difficulties in using the semi-inverse method.

The vector $\boldsymbol{\sigma}_s$ defines the non-self-balanced part of the stress state of the spring. In fact, if $\boldsymbol{\sigma}_s$ is successively multiplied by $\mathbf{a}_m^*$ and integrated over S, we obtain the following relations for the shear forces $Q_{\beta(\xi)}$, the normal force $Q_{3(\xi)}$, the bending moments $M_{\beta(\xi)}$ and the twisting moment $M_{3(\xi)}$

$$d_{11}C_7 + d_{21}C_8 = Q_2, \quad d_{12}C_7 + d_{22}C_8 = Q_3 - M_2 / r_0 \tag{1.17}$$
$$d_{35}C_9 + [d_{35}(\xi - \eta) + d_{55}]C_{11} = r_0^{-1}(iM_1 + M_3 - r_0 Q_2), \quad d_{53}C_{11} = Q_1 - iQ_3$$
$$Q_j = \int \sigma_{j3} dS, \quad M_1 = \int \sigma_{33}\xi_2 dS, \quad M_2 = -\int \sigma_{33}\xi_1 dS, \quad M_3 = \int (\xi_1\sigma_{23} - \xi_2\sigma_{31}) dS$$

If we put $\xi = \eta$ in expressions (1.17), and in the expressions for Q_j and M_j instead of σ_{j3} we substitute their boundary values p_j, we obtain equations for determining the constants C_r (r = 7, 8, 9, 11) and, moreover, $C_{10} = C_9^*$, $C_{12} = C_{11}^*$. Here the elements of the stiffness matrix $d_{lm} = (\mathbf{b}_l, \mathbf{a}_m)$, taking into account the specific form of the vectors $\mathbf{a}_m$ and some specific properties of the scalar products of vectors, from the Jordan chains [4], can be calculated using the following expressions

$$d_{11} = \int b_{21} dS, \quad d_{21} = d_{12} = \int d_{22} dS = r_0^{-1}\int b_{31} r dS, \quad d_{22} = \int b_{32} r dS$$
$$d_{35} = -d_{53} = -r_0^{-1}\int (b_{13}\xi_2 - ib_{33}\xi_2 - b_{23}r) dS = \int (b_{13} - ib_{33}) dS = 2\int b_{13} dS \tag{1.18}$$
$$d_{55} = r_0^{-1}\int (b_{15}\xi_2 - ib_{35}\xi_2 - b_{25}r) dS = -2ir_0^{-1}\int b_{35}\xi_2 dS$$

All the remaining elements are equal to zero.

Note 2. The coefficients C_{l+6} of the Saint-Venant solution (1.16) are defined exactly by relations (1.17) and (1.18). The coefficients C_l can be found exactly only from infinite systems, and approximately by solving the following algebraic system

$$d_{11}C_1 + d_{21}C_2 = g_1, \quad d_{12}C_1 + d_{22}C_2 = g_2, \quad d_{35}C_3 + d_{55}^*C_5 = g_3, \quad d_{53}C_5 = g_5$$
$$C_4 = C_3^*, \quad C_6 = C_5^*; \quad g_m = -\sum_{l=1}^{6} C_{6+l}(\mathbf{u}_{6+l}(-\eta), \mathbf{b}_m)$$

The central problem in the scheme described above is to construct solutions of the four two-dimensional boundary-value problems (1.9)–(1.11), which split naturally into two pairs (within the limits of a pair the problems differ solely in the form of the right-hand sides). The first pair is problem (1.9), (1.10), the solution of which describes the stress–strain state of the spring that is most important in practice: stretching—compression and twisting about its axis. This can easily be seen from relation (1.17) if we take into account that $Q_2(\eta) = Q_z$, $r_0 Q_3(\eta) - M_2(\eta) = M_z$, where Q_z and M_z are the external axial force and external twisting moment, respectively. The second pair is problem (1.11), the solution of which describes, in the general case, a complex stress–strain state, the form of which is revealed be low by an asymptotic analysis of these problems for a spring with a relatively thin wire and a small pitch.

When $Q_r = Q_1(\eta)$ and $Q_\phi = Q_3(\eta)$ are equal to zero, it follows from (1.18) that $C_{11} = C_{12} = 0$ while $\mathrm{Re}C_9 = d_*^{-1}M_r$, $\mathrm{Im}C_9 = d_*^{-1}M_\phi$ and $\mathrm{Im}C_9 = d_*^{-1}M_\phi$, where M_r and M_ϕ are the radial and tangential components of the moment of the external forces about a point on the spring axis $z = h_0\eta$ and

$d_* = -id_{35}$ is a real quantity which follows from the general theory [4]. These moments are non-zero if, for example, the line of action of the force Q_z does not coincide with the spring axis.

All the boundary-value problems (1.9)–(1.11) are self-conjugate and can be reduced, using well-known methods [6], to finding the minima of the quadratic functionals

$$\begin{aligned}
\Psi_m &= \int(W_m + l_m) r dS, \quad m = 1,2,3,5 \\
W_m &= \theta|\psi_m|^2 + (|\beta_{11m}|^2 + |\beta_{22m}|^2 + |\beta_{33m}|^2) + 2(|\beta_{12m}|^2 + |\beta_{13m}|^2 + |\beta_{23m}|^2) \\
\psi_m &= \beta_{11m} + \beta_{22m} + \beta_{33m}
\end{aligned} \tag{1.19}$$

Expressions for β_{ijm} are obtained from (1.3) for ε_{ij} by replacing the component u_i in them by a_{i6+m} and the operator D by the operators D_m from the group of formulae (1.15)

$$\begin{aligned}
l_1 &= 4\beta_{131}, \quad l_2 = 2(\theta\psi_2 + \beta_{332}) r / r_0 \\
l_3 &= (2r[\mathrm{Im}(\theta\psi_3 + \beta_{333}) + 2\,\mathrm{Re}\,\beta_{133}]\xi_2 + 4\,\mathrm{Re}(\xi_2\beta_{133} - r\beta_{233}))/(rr_0^2) \\
l_5 &= 2\,\mathrm{Re}(\theta\psi_5 a_{39}^* + \beta_{225} a_{39}^* + 2\beta_{135} a_{19}^* + 2\beta_{235} a_{29}^*)
\end{aligned} \tag{1.20}$$

Note 3. Since the homogeneous problem (1.9) has non-trivial solutions $\mathbf{a}_1$ and $\mathbf{a}_2$, while the homogeneous problem (1.11) has the non-trivial solution $\mathbf{a}_3$, the variational problems (1.19) are uniquely solvable with the additional conditions

with $m = 1, 2$

$$\int(\mathbf{a}_{m+6} * \mathbf{a}_1) dS = \int a_{2m+6} dS = 0, \quad \int(\mathbf{a}_{m+6} * \mathbf{a}_2) dS = \int a_{3m+6} r dS = 0 \tag{1.21}$$

with $m = 3, 5$

$$\int(\mathbf{a}_{m+6} * \mathbf{a}_3^*) dS = \int(a_{1m+6} - i a_{3m+6}) dS = 0 \tag{1.22}$$

2. THE ASYMPTOTIC SOLUTION

Without going into detail on the analysis, we will explain the main ideas of the approach and give the final results.

We will denote the characteristic linear dimension of the section S by a and we will introduce the following dimensionless parameters and coordinates for the spring

$$\varepsilon = a / r_0, \quad \beta = h_0 / a, \quad \alpha_1 = \xi_1 / a, \quad \alpha_2 = \xi_2 / a \tag{2.1}$$

Converting the equations and boundary conditions (1.9)–(1.11), taking (2.1) into account, we obtain boundary-value problems with the parameter ε. We will use ε as a small parameter and we will assume that $\beta = 0(1)$ with respect to ε. Seeking a solution in the form of series in a small parameter, we obtain a recurrent system of plane and antiplane problems of the theory of elasticity with different right-hand sides, the form of which depends on the order of the approximation. For an arbitrary section S, analytic solutions can be constructed only in low approximations, and only the first terms of the asymptotic expansions for the stresses and the coefficients of the stiffness matrix can be obtained in explicit form. Similar solutions can be obtained using the classical theory of curvilinear rods, which is based on the hypothesis of plane sections. The main difficulties are due to the construction of solutions for plane problems, which even for elliptical and rectangular cross-sections have no exact analytic solutions. The use of numerical methods, in particular the method of finite elements (MFE) to solve them is hardly convenient since it is equivalent in costs to the solution of the initial problems. When S is a circle, explicit analytic expressions can be obtained in high approximations.

Below we give asymptotic expansions for the coefficients of the stiffness and stress matrix, which enable us to take into account the effect both of the curvature (the parameter ε), and the pitch of the spring (the parameter β) on the stress–strain state. The refined formulae, which enable us to take into account the effect of the curvature on the maximum tangential stresses in the case of the stretching-compression of a spring, were apparently obtained for the first time by Timoshenko [7] and then refined [8–10].

We will confine ourselves to the important practical case when $Q_r = Q_\phi = 0$. We have

$$d_{11}=\varepsilon^3\mu\lambda_{22}B_1,\quad d_{12}=-\varepsilon^4\mu\lambda_{12}B_1,\quad d_{22}=\varepsilon^3\mu\lambda_{11}B_2,\quad d_*=-id_{35}=\varepsilon^4 B_3 \tag{2.2}$$

$$\sigma_{ij}=Q^{(1)}\sigma_{ij}^{(1)}+Q^{(2)}\sigma_{ij}^{(2)}+Q^{(3)}(\sigma_{ij}^{(3)}\cos\phi-\sigma_{ij}^{(3)}\sin\phi)+Q^{(4)}(\sigma_{ij}^{(4)}\cos\phi+\sigma_{ij}^{(3)}\sin\phi) \tag{2.3}$$

$$Q^{(1)}=\varepsilon^{-3}Q_z r_0^{-1}aB_1\Omega,\quad Q^{(2)}=\varepsilon^{-3}M_z r_0^{-2}aB_2\Omega$$
$$Q^{(3)}=\varepsilon^{-4}M_r r_0^{-2}aB_3,\quad Q^{(4)}=\varepsilon^{-4}M_\phi r_0^{-2}aB_3$$
$$\sigma_{31}^{(1)}=\lambda_{11}f_{31}^{(1)}+\varepsilon^2\lambda_{12}f_{31}^{(2)},\quad \sigma_{32}^{(1)}=\lambda_{11}f_{32}^{(1)}+\varepsilon^2\lambda_{12}f_{32}^{(2)}$$
$$\sigma_{33}^{(1)}=\varepsilon\lambda_{11}f_{33}^{(1)}+\varepsilon\lambda_{12}f_{33}^{(2)},\quad \sigma_{\alpha\beta}^{(1)}=O(\varepsilon^2)$$
$$\sigma_{31}^{(2)}=\varepsilon(\lambda_{21}f_{31}^{(1)}+\lambda_{22}f_{31}^{(2)}),\quad \sigma_{32}^{(2)}=\varepsilon(\lambda_{21}f_{32}^{(1)}+\lambda_{22}f_{32}^{(2)})$$
$$\sigma_{33}^{(2)}=\lambda_{22}f_{33}^{(2)}+\varepsilon^2\lambda_{21}f_{33}^{(2)},\quad \sigma_{\alpha\beta}^{(2)}=O(\varepsilon^2)$$
$$f_{31}^{(1)}=\alpha_2-\varepsilon 5\alpha_1\alpha_2/4+\varepsilon^2\alpha_1^2\alpha_2/4,\quad f_{31}^{(2)}=\beta(\alpha_2-\varepsilon\alpha_1\alpha_2)$$
$$f_{32}^{(1)}=-\alpha_1+\varepsilon(3+7\alpha_1^2-3\alpha_2^2)/8+\varepsilon^2(-3\alpha_1+\alpha_1^3+3\alpha_1\alpha_2^2)/8$$
$$f_{32}^{(2)}=\beta(1+2\nu)\alpha_2,\quad f_{33}^{(1)}=2\beta\nu(-\alpha_1+\varepsilon\alpha_1^2)$$
$$f_{33}^{(2)}=2(1+\nu)\alpha_1+\varepsilon[-\alpha_1^2(4+5\nu+2\nu^2)+2\nu(3+2\nu)\alpha_2^2-\nu(3+2\nu)]/4+$$
$$+\nu\varepsilon^2[2(1+2\nu)\alpha_1^3-2(3+2\nu)\alpha_1\alpha_2^2+(3+2\nu)\alpha_1]/4$$
$$\lambda_{11}=1+\varepsilon^2(8+13\nu+6\nu^2)/[24(1+\nu)]$$

$$\lambda_{12}=\beta\nu/(1+\nu),\quad \lambda_{21}=\beta\nu,\quad \lambda_{22}=1+(23/48)\varepsilon^2,\quad \Omega=\lambda_{11}\lambda_{22}-\varepsilon^2\lambda_{12}\lambda_{21}$$
$$\sigma_{31}^{(3)}=\beta^{-1}[2k\alpha_2+\varepsilon(\nu-\nu k+3k/4),\quad \sigma_{32}^{(3)}=\beta^{-1}\{-2k\alpha_1+$$
$$+\varepsilon[(5k+5k\nu-6\nu)\alpha_1^2/2+(3k+2k\nu)\alpha_2^2/4+(2\nu-3k-2k\nu)/4]\}$$
$$\sigma_{33}^{(3)}=\varepsilon k\beta^{-1}[4\theta\nu(\alpha_1^2-\alpha_2^2)+2(1+2k+\nu)\alpha_1-2(1+\nu)]$$
$$\sigma_{11}^{(3)}=\varepsilon\beta^{-1}[k/(1+2\nu)-4k\theta\alpha_2^2-4\nu\alpha_1]$$
$$\sigma_{22}^{(3)}=\varepsilon\beta^{-1}[-k/(1+2\nu)+4k\theta\alpha_1^2-4\nu\alpha_1-4k\beta]$$
$$\sigma_{12}^{(3)}=\varepsilon 2(k+2\nu\beta^{-1})\alpha_2$$
$$\sigma_{31}^{(4)}=O(\varepsilon^2),\quad \sigma_{32}^{(4)}=\varepsilon 4k\alpha_2,\quad \sigma_{33}^{(4)}=4\beta^{-1}\alpha_2[k+\varepsilon 4(1+\nu)]$$
$$\sigma_{11}^{(4)}=-\varepsilon 4\nu\beta^{-1}\alpha_2,\quad \sigma_{22}^{(4)}=\varepsilon 4\beta^{-1}(-4k\theta\alpha_1\alpha_2+4\nu\alpha_2)$$
$$\sigma_{12}^{(4)}=\varepsilon\beta^{-1}[k(\alpha_1^2+\alpha_2^2)/(1-2\nu)-4\nu\alpha_1]$$
$$B_1=a^{-4}C,\quad B_2=a^{-4}B_{22},\quad B_3=4a^{-4}\beta^{-1}CB_{11}/(C+B_{11}),\quad k=B_{11}/(C+B_{11}) \tag{2.4}$$
$$B_{\alpha\alpha}=2(1+\nu)\int\xi_\alpha^2\,dS$$

In (2.4) μC is the stiffness to twisting of a prismatic rod with cross-section S.

Expressions in the form (2.2) and (2.3) in the principal terms of the asymptotic expansions can be used for a spring with an arbitrary wire cross-section.

When the region S is a circle of radius A, the expressions for B_j and k have the simple form

$$B_1=\pi/2,\quad B_2=\pi(1+\nu)/2,\quad B_3=\pi k,\quad k=(1+\nu)/(2+\nu)$$

However, if the cross-section, orthogonal to the wire axis of the spring, is a circle (which corresponds more to reality), the region S will be an ellipse with semiaxes $a_1=a$, $a_2=a(1+\varepsilon^2\beta^2)^{1/2}$ (this follows from simple geometrical considerations: $a_2=a/\cos\alpha$, $\mathrm{tg}\,\alpha=h/(2\pi r_0)=\varepsilon\beta$) and we must use formulae (2.4), putting

$$C = \pi a^4 (1+\varepsilon^2\beta^2)^{3/2} / (2+\varepsilon^2\beta^2) \tag{2.5}$$

$$B_{11} = \pi(1+\nu)(1+\varepsilon^2\beta^2)^{1/2}/2, \quad B_{22} = \pi(1+\nu)(1+\varepsilon^2\beta^2)^{3/2}/2$$

in them.

In this case the error in calculating the main stresses due to the force Q_z and the moment M_z, according to the above formulae, will have an asymptotic error of $O(\varepsilon^3)$.

We will present further refined formulae for the characteristics of a spring of most practical importance [7, 11], namely for its settlement $\Delta = u_2(\eta)$ and the maximum shear stresses $\tau_{\max}$, due to the axial force. We have, with an asymptotic error of $O(\varepsilon^3)$

$$\Delta = \frac{nQ_z}{\mu r_0 \varepsilon^2}\delta_1(\varepsilon,\alpha), \quad \tau_{\max} = \frac{Q_z}{\pi a^2\varepsilon^2}\delta_2(\varepsilon,\alpha)$$

$$\delta_1(\varepsilon,\alpha) = X(\lambda_{11} - \varepsilon\lambda_{12}), \quad \delta_2(\varepsilon,\alpha) = X\lambda_{11}(1 + \tfrac{5}{4}\varepsilon + \tfrac{1}{2}\varepsilon^2) \tag{2.6}$$

$$X = \Omega^{-1}(2+\mathrm{tg}^2\,\alpha)(1+\mathrm{tg}^2\,\alpha)^{-3/2}$$

3. RESULTS OF A NUMERICAL ANALYSIS

We carried out a number of calculations for a spring with an elliptical and a rectangular cross-section. In the case of an elliptical cross-section, using (2.6) we analysed the behaviour of the spring settlement and the maximum shear stresses as a function of the parameters ε and α. The results are shown in Fig. 1 in the form of a graph of $\delta_1(\varepsilon, \alpha)$ (the dashed curves) and $\delta_2(\varepsilon, \alpha)$ (the continuous curves).

In the case of a rectangular cross-section the calculations were carried out using the method of finite elements (MFE). We chose as the finite element a rectangle with four nodal points, and solutions of the problems of the stretching and twisting of a spring (1.19) and (1.21) were constructed as linear combinations of the unknown nodal displacements and bilinear basis functions.

The scheme for obtaining a system of algebraic equations from the variational equation, which is a consequence of (1.19), for the whole finite element grid, is standard [12]. However, a specific feature of the problems considered ($\gamma = 0$ is an eigenvalue) is such that the system obtained is degenerate, so that the rank of a system of order n is equal to $n - 2$. The arbitrariness in the solution of such a system is removed using two additional conditions (1.21), in which case we need to take into account that the matrix ceases to be a band matrix and is symmetrical.

The problems considered are illustrated by numerical calculations (MFE) and using asymptotic formulae (AF), which enables us to indicate the limits of applicability of the asymptotic form. Table 1 shows the stiffness d_{ii}, calculated, apart from the factors μ and r_0 and orders of k_p, for a constant value of $\beta = 0.15916$ for a square cross-section.

According to the results presented, the difference between the numerical and asymptotic data for d_{11} with $\varepsilon < 0.09$ does not exceed 2%, while for d_{22} it does not exceed 5%. The asymptotic formulae give an error greater than 10% with $\varepsilon \geqslant 0.5$.

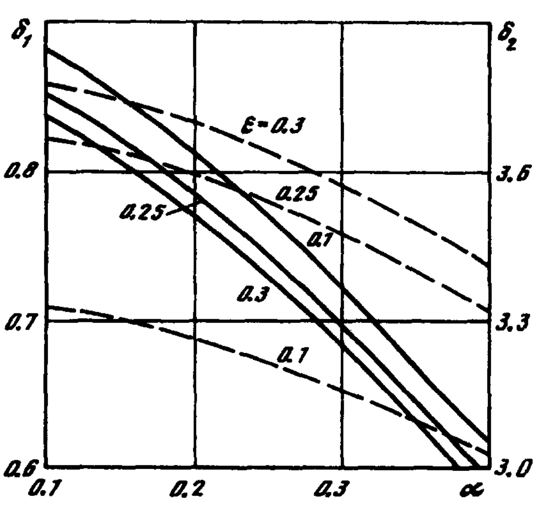

Fig. 1.

Table 1

ε	d_{11}		d_{22}		k_p
	MFE	AF	MFE	AF	
0.01	1.425	1.400	2.664	2.759	10^{-5}
0.03	1.282	1.260	2.396	2.473	10^{-4}
0.05	3.562	3.500	6.653	6.897	10^{-4}
0.07	6.984	6.860	13.030	13.842	10^{-4}
0.09	1.154	1.134	2.152	2.234	10^{-3}
0.1	1.425	1.400	2.655	2.759	10^{-3}
0.3	1.289	1.260	2.362	2.473	10^{-2}
0.5	3.617	3.500	6.453	6.897	10^{-2}
0.7	7.208	6.860	1.238	1.384	10^{-1}
0.9	1.221	1.400	1.990	2.234	10^{-1}

Table 2

ε	α	a_2/a_1	d_{11}	d_{21}	d_{22}
0.5	π/15	1.0	0.03514	−0.002879	0.0612
0.5	π/10	1.08	0.03368	−0.004074	0.0566
0.5	π/4	1.4	0.03812	−0.000461	0.3736
0.9	π/15	1.0	11.97	−0.009543	19.28
0.9	π/10	1.08	11.40	−0.013787	17.83
0.9	π/4	1.4	11.86	−0.003426	11.90

Table 2 illustrates the stiffness as a function of the density of the turns, where ε is fixed in an interval inaccessible for an asymptotic form. The density of the turns is characterized by tgα, and its change for $\alpha \neq 0$ leads to a consideration of a rectangle as the cross-section and not a square.

The analysis of the convergence of the data as a function of the splitting is due to the choice of the specific splittings of 8 × 14 and 8 × 8 for a rectangle and a square respectively.

This research was supported by the Russian Foundation for Basic Research (97-01-00-464).

REFERENCES

1. USTINOV, Yu. A., The Saint-Venant problem for a spring. *Dokl. Ross. Akad. Nauk*, 1995, **345**, 5, 621–623.
2. LUR'YE, A. I., *The Theory of Elasticity*. Nauka, Moscow, 1970.
3. DRUZ', A. N., POLYAKOV, N. A. and USTINOV, Yu. A., Homogeneous solutions and Saint-Venant problems for a naturally twisted rod. *Prikl. Mat. Mekh.*, 1996, **60**, 4, 660–668.
4. GETMAN, I. P. and USTINOV, Yu. A., *The Mathematical Theory of Solid Irregular Waveguides*. Izd. Rost. Univ. Rostov-on-Don, 1993.
5. DRUZ', A. N. and USTINOV, Yu. A., Green's tensor for an elastic cylinder and its applications to the development of the St Venant theory. *Prikl. Mat. Mekh.*, 1996, **60**, 1, 102–110.
6. MIKHLIN, S. G., *Variational Methods in Mathematical Physics*. Nauka, Moscow, 1970.
7. TIMOSHENKO, S. P., *Strength of Materials*. P.1. *Elementary Theory and Problems*, 3rd edition. Van Nostrand, Princeton, NJ, 1955.
8. ANCKER, C. J. and GOODIER, J. N., Pitch and curvature corrections for the helical springs. *J. Appl. Mech.*, 1958, **25**, 4, 466–470.
9. ANCKER, C. J. and GOODIER, J. N., Theory of pitch and curvature corrections for the helical springs—I (tension). *J. Appl. Mech.*, 1958, **25**, 4, 471–483.
10. ANCKER, C. J. and GOODIER, J. N., Theory of pitch and curvature corrections for the helical springs—II (tension). *J. Appl. Mech.*, 1958, **25**, 4, 484–495.
11. ORLOV, P. I., *Fundamentals of Construction*. Vol. 2. Mashinostroyeniye, Moscow, 1988.
12. ZIENKIEWICZ, O. and MORGAN, K., *Finite Elements and Approximation*. Wiley, New York, 1983.

Translated by R.C.G.